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THE GRAHAM CONJECTURE IMPLIES THE ERDŐS-TURÁN CONJECTURE

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Abstract: Erdős and Turán once conjectured that any set \( A \subset N \) with \( \sum_{n \in A} 1/n = \infty \) should contain infinitely many progressions of arbitrary length \( k \geq 3 \). For the two-dimensional case, Graham conjectured that if \( B \subset N \times N \) satisfies
\[
\sum_{(x,y) \in B} \frac{1}{x^2 + y^2} = \infty,
\]
then for any \( x \geq 2 \), \( B \) contains an \( x \times x \) non-parallel grid. In this paper it is shown that if the Graham conjecture is true for some \( x \geq 2 \), then the Erdős-Turán conjecture is true for \( k = 2x - 2 \).

1. Introduction

One famous conjecture of Erdős and Turán [2] asserts that any set \( A \subset N \) with \( \sum_{n \in A} 1/n = \infty \) should contain infinitely many progressions of arbitrary length \( k \geq 3 \). There are two important progresses towards this direction due to Szemerédi [7] and Green and Tao [6], respectively, which assert that if \( A \) has positive upper density or \( A \) is in the set of the prime numbers, then \( A \) contains infinitely many progressions of arbitrary length.

If one considers the similar question in the two-dimensional plane, Graham [3] conjectured that if \( B \subset N \times N \) satisfies
\[
\sum_{(x,y) \in B} \frac{1}{x^2 + y^2} = \infty,
\]
then \( B \) contains the four vertices of an non-parallel square. More generally, for any \( k \geq 2 \) it should be true that \( B \) contains an \( n \times n \) non-parallel grid. Furstenberg and Katznelson [3] proved the two-dimensional Szemerédi theorem, that is, any set \( B \subset N \times N \) with positive upper density contains an \( n \times n \) non-parallel grid. In another words, such a set \( B \) contains any finite pattern.

The purpose of this paper is to show that if the Graham conjecture is true, then the Erdős-Turán conjecture is also true.

2. The Graham Conjecture Implies the Erdős-Turan Conjecture

Suppose that the Erdős-Turán conjecture is false for \( k = 3 \). Then there exists a set
\[
A = \{a_1 < a_2 < a_3 < \ldots \} \subset N
\]
with $\sum_{i=1}^{m} 1/a_i = \infty$ such that $A$ contains no arithmetic progression of length 3. Define a set $B \in \mathbb{N} \times \mathbb{N}$ by

$$B = \left\{ (a_n, m, n) : n \in \mathbb{N}, m \in \mathbb{N} \right\}.$$ 

Then

$$\sum_{i=1}^{m} 1/a_i^2 \geq \sum_{i=1}^{m} \frac{1}{|a_i - m|^2 + m^2} \geq \sum_{i=1}^{m} \frac{1}{|a_i - m|^2} \geq \sum_{i=1}^{m} \frac{1}{|a_i - a_{i+1}|^2} = \sum_{i=1}^{m} \frac{1}{b_{i+1}} = \infty.$$ 

In the sequel we indicate that $B$ contains no square and argue by contradiction. This would mean that the Graham conjecture is false for $x = 3$. Suppose that for some $n, m, l \in \mathbb{N}$, $B$ contains a square of the following form:

$$(a_n - m, m + l), (a_n - m + l, m + l - 1), (m, m + l - m), (m + l, m + l).$$

It follows easily from the construction of $B$ that $a_{n+1} - 1, a_{n+1} + l \in A$, which yields a contraction since $A$ contains no arithmetic progression of length 3 according to the initial assumption.

Similarly, if the Graham conjecture is true for some $x \geq 2$, then the Erdős-Turán conjecture is true for $k = 2x - 1$. The interested reader can easily provide a proof.

3. CONCLUDING REMARKS

Let $r(k, N)$ be the maximal cardinality of a subset $A$ of $\{1, 2, \ldots, N\}$ which is free of $k$-term arithmetic progressions. Behrend [1] and Rankin [6] had shown that $r(k, N) \sim C k \cdot N^{2/k}$. Thus

$$\sum_{k=1}^{N} r(k, N) \geq C N^{2} \sum_{k=1}^{N} k^{-(2/k)} = \infty.$$
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